

HIGH GIRTH HYPERGRAPHS WITH UNAVOIDABLE MONOCHROMATIC OR RAINBOW EDGES

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ABSTRACT. A classical result of Erdős and Hajnal claims that for any integers $k, r, g \geq 2$ there is an r -uniform hypergraph of girth at least g with chromatic number at least k . This implies that there are sparse hypergraphs such that in any coloring of their vertices with at most $k - 1$ colors there is a monochromatic hyperedge. We show that for any integers $r, g \geq 2$ there is an r -uniform hypergraph of girth at least g such that in any coloring of its vertices there is either a monochromatic or a rainbow (totally multicolored) edge. We give a probabilistic and a deterministic proof of this result.

1. INTRODUCTION

A classical result of Erdős and Hajnal [4], Corollary 13.4, claims that for any integers $k, r, g \geq 2$ there is an r -uniform hypergraph of girth at least g with chromatic number at least k . This implies that there are sparse hypergraphs such that in any coloring of their vertices with at most $k - 1$ colors there is a monochromatic hyperedge. The original proof was probabilistic. Other probabilistic constructions were given by Nešetřil and Rödl [16], Duffus et al. [2], Kostochka and Rödl [9], and, in case of graphs only, by Erdős [3]. Several explicit constructions were found later, see Lovász [12], Erdős and Lovász [5], Nešetřil and Rödl [17], Duffus et al. [2], Alon et al. [1], Kríž [10], Kostochka and Nešetřil [8]. Nešetřil [15] as well as Raigorodskii and Shabanov [18] gave surveys on the topic. Some interesting generalizations were treated by Feder and Vardi [6], Kun [11], Müller [13], [14], as well as by Nešetřil [15].

When the number of colors used on the vertices of a hypergraph is not restricted, the monochromatic hyperedges could easily be avoided by simply using a lot of different colors. Then, however, so-called rainbow (totally multicolored) hyperedges could appear. The notion of a proper coloring when both rainbow and monochromatic hyperedges are forbidden was introduced by Voloshin in a concept called bihypergraphs, [19], see also Karrer [7]. Here, we show that there are sparse hypergraphs in which monochromatic or rainbow hyperedges are unavoidable.

A *cycle* of length g in a hypergraph is a subhypergraph consisting of $g \geq 2$ distinct hyperedges E_0, \dots, E_{g-1} and containing distinct vertices x_0, \dots, x_{g-1} , such that $x_i \in E_i \cap E_{i+1}$, $i = 0, \dots, g - 1$, addition of indices modulo g . The *girth* of a hypergraph is the length of a shortest cycle if such exists, and infinity otherwise. Next is our main result.

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Theorem 1. *For any integers $r, g \geq 2$ there is an r -uniform hypergraph of girth at least g such that in any coloring of its vertices there is either a monochromatic or a rainbow (totally multicolored) edge.*

We shall give a probabilistic proof and an explicit construction of a desired hypergraph. Our proofs are inspired by amalgamation and probabilistic techniques of Nešetřil and Rödl. To shorten the presentation, we shall say that a hypergraph is *rm-unavoidable* if any coloring of its vertices has either a rainbow or a monochromatic edge. We give an explicit construction and use it to prove the main theorem in Section 2. The probabilistic proof is given in Section 3. The proofs of a few standard results we use are presented in Appendix.

2. EXPLICIT CONSTRUCTION OF RM-UNAVOIDABLE HYPERGRAPHS

The goal of this section is to construct, for each $r \geq 2$ and $g \geq 2$, an rm-unavoidable hypergraph, that we shall call $H(r, g)$, of uniformity r and girth g . The three main concepts we use are amalgamation, special partite hypergraphs forcing rainbow edges, and so-called complete partite factors. All of these notions are defined for partite hypergraphs. A hypergraph is *a-partite* if its vertex set can be partitioned in at most a parts such that each hyperedge contains at most one vertex from each part. We shall first define a part-rainbow-forced hypergraph as a hypergraph having some special coloring properties and give an explicit construction of such a hypergraph $PR(r, g)$. Then we incorporate this hypergraph into a more involved construction of an rm-unavoidable hypergraph $H(r, g)$. Both of these constructions use amalgamation.

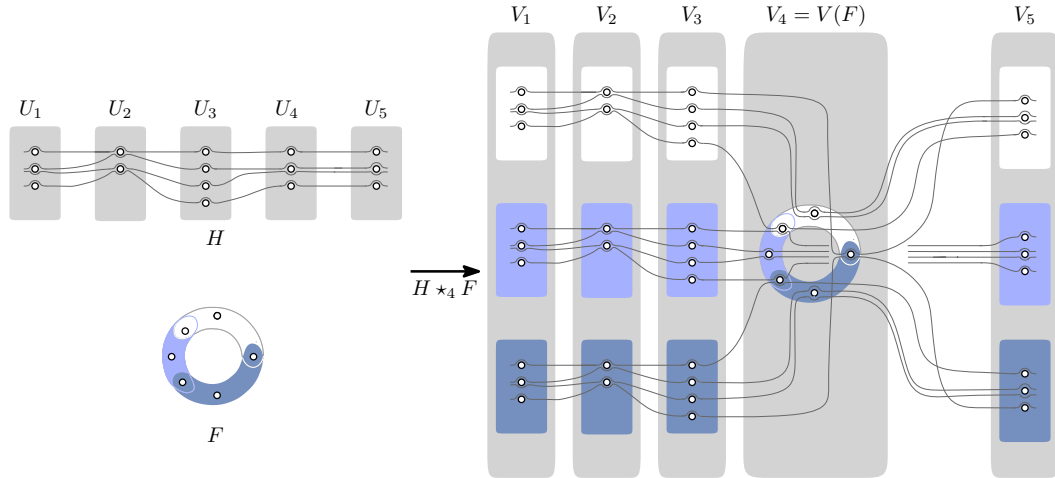


FIGURE 1. Amalgamation of F and H along the 4th part. Here F is a 3-uniform cycle on 3 edges, H is 5-uniform, 5-partite with 4 edges. The resulting graph is 5-partite, 5-uniform, with curves indicating hyperedges and colors indicating distinct copies of H , corresponding to the edges of F .

Amalgamation: Given an a -partite hypergraph H with the i^{th} part of size r_i and given an r_i -uniform hypergraph $F = (V, \mathcal{E})$, an *amalgamation* of H and F along the i^{th} part, denoted by $H \star_i F$ is an a -partite hypergraph obtained by taking $|\mathcal{E}|$ vertex-disjoint copies of H and identifying the i th part of each such copy with a hyperedge of F such that distinct copies get identified with distinct hyperedges. Moreover, the j^{th} part of $H \star_i F$ is a pairwise disjoint union of the j^{th} parts from the copies of H , for $j \in \{1, \dots, a\} \setminus \{i\}$, see Figure 1. We shall sometimes say that $H \star_i F$ is obtained by amalgamating copies of H along the part i using F .

Part-rainbow-forced hypergraph: A vertex coloring of an a -partite hypergraph with parts X_1, \dots, X_a that assign $|X_i|$ colors to part i , $i = 1, \dots, a$ is called *part rainbow*. We say that an a -partite hypergraph is *part-rainbow-forced* if in any part-rainbow coloring there is a rainbow edge.

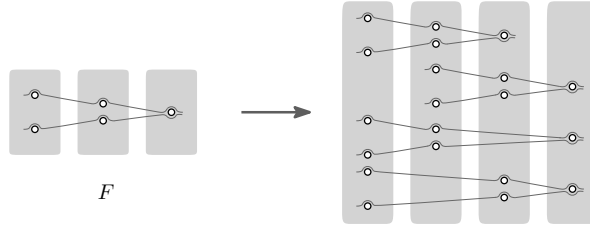


FIGURE 2. An example of a complete 4-partite F -factor, where F is a 3-partite 3-uniform hypergraph with two edges.

Partite factor: Let F be an r -uniform r -partite hypergraph. A *complete a -partite F -factor* is an a -partite r -uniform hypergraph G that is a union of pairwise vertex-disjoint copies $F_1, \dots, F_{\binom{a}{r}}$ of F , such that each part of F_i is contained in some part of G , $i = 1, \dots, \binom{a}{r}$ and such that the union of any r parts of G contains the vertex set of F_i , for some $i = 1, \dots, \binom{a}{r}$, see Figure 2.

Construction of a hypergraph $PR(r, g)$: Let $r, g \geq 2$, $g \geq 2$ be fixed. Let $g \geq 2$, let $PR(2, g)$ be a bipartite graph on vertices x, y, z and edges xy, yz .

Assume now that $PR(r, g)$ has been constructed and it is an r -uniform, r -partite hypergraph. Let F' be an ℓ -uniform hypergraph of girth at least g and minimum degree $\ell(r+1)$, where $\ell = |E(H_r)|$. We show the existence of F' in Appendix.

For an r -uniform r -partite hypergraph H , let \tilde{H} be an $(r+1)$ -partite $(r+1)$ -uniform hypergraph that is obtained from H by expanding each of its edges by a vertex in a new, $(r+1)^{\text{st}}$ part such that each edge is extended by an own vertex, i.e., the size of the $(r+1)^{\text{st}}$ part is equal to the number of edges in H , see Figure 3.

Let $\widetilde{PR(r, g)} = \widetilde{PR(r, g)} \star_{r+1} F'$, i.e., it is an amalgamation of copies of $\widetilde{PR(r, g)}$ along the $(r+1)^{\text{st}}$ part using F' , see Figure 4.

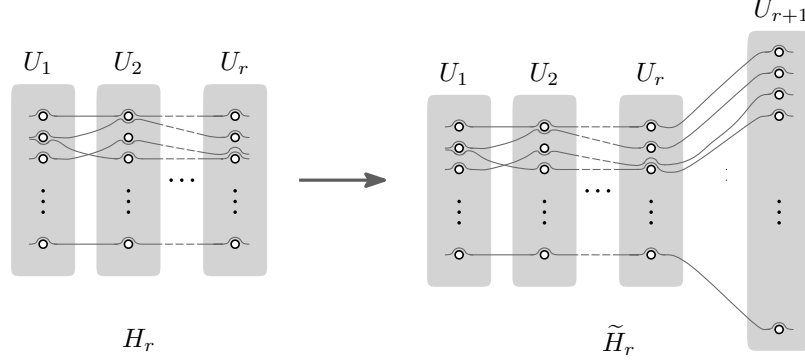


FIGURE 3. Extension of an r -partite r -uniform hypergraph H_r to an $(r+1)$ -partite $(r+1)$ -uniform hypergraph \tilde{H}_r .

Lemma 2. *For any integers $r, g \geq 2$, $PR(r, g)$ is a part-rainbow-forced r -uniform hypergraph of girth g .*

Proof. By construction, $PR(r, g)$ is an r -uniform r -partite hypergraph, $r \geq 2$. We shall prove by induction on r that $PR(r, g)$ is part-rainbow-forced hypergraph of girth at least g . When $r = 2$, we see that a part-rainbow coloring assigns distinct colors to x and z . Thus, no matter how y is colored, xy or yz is rainbow. Moreover this graph is acyclic, so it has infinite girth.

Assume that $PR(r, g)$ is part-rainbow-forced hypergraph of girth at least g . Let's prove that $H_{r+1} = PR(r+1, g)$ is also part-rainbow-forced hypergraph of girth at least g . Let $H_r = PR(r, g)$. Recall that H_{r+1} is an amalgamation of copies $\tilde{H}_r^1, \tilde{H}_r^2, \dots, \tilde{H}_r^{e'}$ of \tilde{H}_r along the $(r+1)^{\text{st}}$ part using F' , where F' is an ℓ -uniform hypergraph of girth at least g , minimum degree $\ell(r+1)$, $\ell = |E(H_r)|$, and $e' = |E(F')|$. Recall further, that \tilde{H}_r^i is obtained by an extension operation tilde from H_r^i , a copy of H_r .

First we shall verify that any part-rainbow coloring c of H_{r+1} results in a rainbow edge. For any $i = 1, \dots, e'$, consider a restriction of c to the vertex set of H_r^i . Since it is a copy of $H_r = PR(r, g)$, it is again part-rainbow, so there is a rainbow edge E'_i in that copy. Let $E'_i \cup \{v_i\}$ be a corresponding uniquely defined edge of \tilde{H}_r^i . The vertices $v_1, \dots, v_{e'}$ are vertices of F' . Since the minimum degree of F' is at least $\ell(r+1)$, then $e' = |E(F')| \geq |V(F')|\ell(r+1)/\ell = |V(F')|(r+1)$. Thus there are at least $r+1$ repeated vertices in the list $v_1, \dots, v_{e'}$, i.e., w.l.o.g. $v = v_1 = \dots = v_{r+1}$. Thus v extends rainbow edges $E'_1, E'_2, \dots, E'_{r+1}$ in $H_r^1, H_r^2, \dots, H_r^{r+1}$. We claim that at least one of the extended edges $E'_1 \cup \{v\}, E'_2 \cup \{v\}, \dots, E'_{r+1} \cup \{v\}$ is rainbow. Assume not, then $c(v)$ is present in each of $E'_1, E'_2, \dots, E'_{r+1}$. However, there are at most r vertices of each given color in the first r parts. Since $E'_1, E'_2, \dots, E'_{r+1}$ are pairwise disjoint, we have a contradiction.

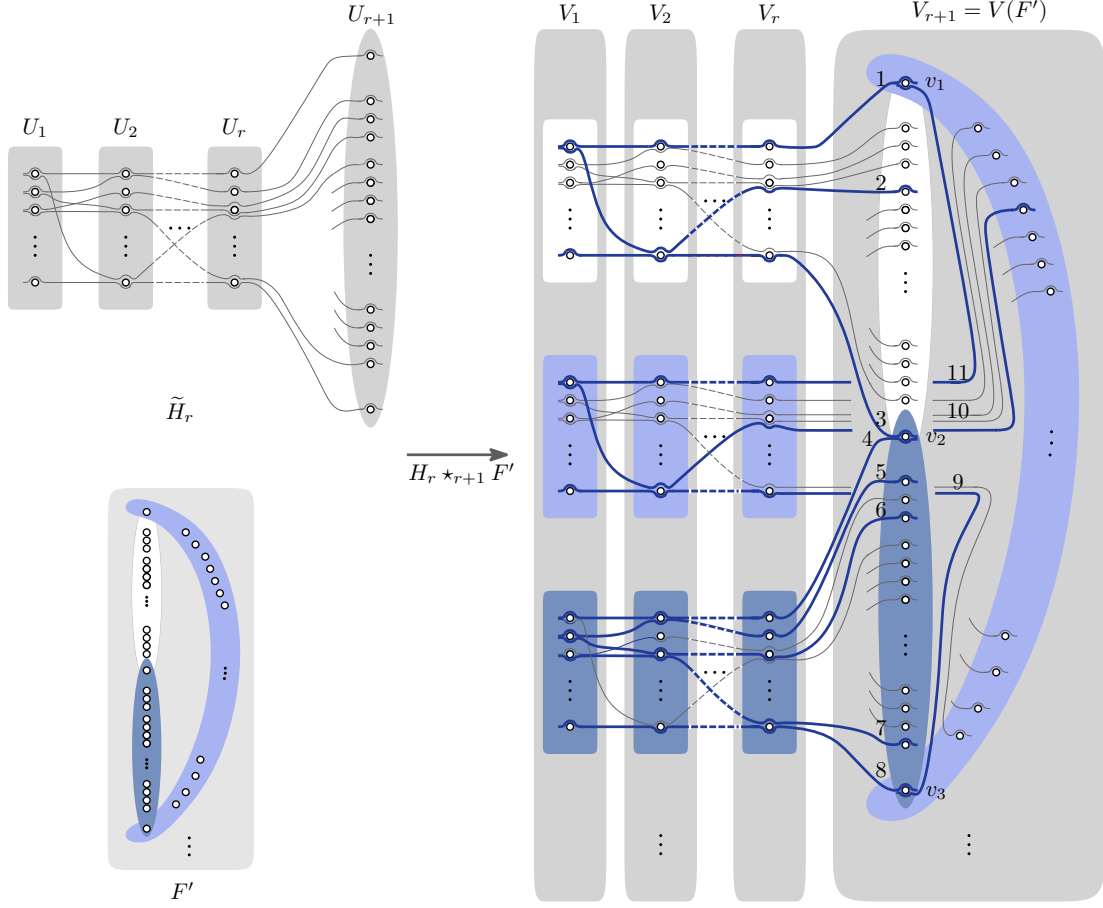


FIGURE 4. Illustration of a part-rainbow-forced $(r+1)$ -uniform hypergraph and a cycle of length 3 in the amalgamated hypergraph F' . The bold hyperedges form a cycle of length 11 in the resulted hypergraph.

To see that the girth of H_{r+1} is at least g , consider a cycle C in H_{r+1} , see bold edges in Figure 4. If the edges of C come from one copy of \tilde{H}_r , then the length of C is at least g as the girth of \tilde{H}_r is the same as girth of H_r . If the edges of C come from at least two distinct copies of \tilde{H}_r , then C is a union of hyperpaths P_0, P_1, \dots, P_{m-1} from different copies of \tilde{H}_r , such that the consecutive paths share a vertex in the last $(r+1)^{\text{st}}$ part, i.e., $V(P_i) \cap V(P_{i+1}) = \{u_i\}$, u_0, \dots, u_{m-1} are distinct vertices from V_{r+1} , addition modulo m . Thus u_i and u_{i+1} belong to the same copy of \tilde{H}_r and thus the same edge of F' , $i = 0, \dots, m-1$, addition modulo m . We see that these edges of F' form a cycle in F' of length at most the length of C . On the other hand, we know that any cycle in F' has length at least g , implying that C has length at least g . This concludes the proof that $PR(r+1, g)$ is part-rainbow-forced of girth at least g . \square

Now we construct an rm-unavoidable hypergraph $H(r, g)$ of uniformity r and girth at least g .

Construction of a hypergraph $H(r, g)$: For $g = 2$ and any $r \geq 2$, let $H(r, 2)$ be a complete r -uniform hypergraph on $(r - 1)^2 + 1$ vertices. Assume that for any $r \geq 2$, $H(r, g - 1)$ has been constructed. Let $F = PR(r, g)$ be as given in the previous construction. Let $a = (r - 1)^2 + r$ and let \mathcal{M}_1 be a complete a -partite F -factor. For any partite hypergraph G , let $|G|_i$ denote the size of the i^{th} part of G . Let $\mathcal{M}_2 = \mathcal{M}_1 \star_1 \mathcal{H}_1$, where $\mathcal{H}_1 = H(|\mathcal{M}_1|_1, g - 1)$. Let $\mathcal{M}_3 = \mathcal{M}_2 \star_2 \mathcal{H}_2$, where $\mathcal{H}_2 = H(|\mathcal{M}_2|_2, g - 1)$. In general, let $\mathcal{M}_{j+1} = \mathcal{M}_j \star_j \mathcal{H}_j$, where $\mathcal{H}_j = H(|\mathcal{M}_j|_j, g - 1)$. We see that the j^{th} part of \mathcal{M}_{j+1} corresponds to the vertex set of \mathcal{H}_j . Let $H(r, g) = \mathcal{M}_{a+1}$.

Now, we shall prove that this construction gives an rm-unavoidable hypergraph that is r -uniform and has girth g . This will give a proof of Theorem 1.

Proof of Theorem 1. We shall show that $H(r, g)$ is an rm-unavoidable hypergraph of girth at least g , by induction on g . When $g = 2$, $H(r, 2)$ is a complete r -uniform hypergraph on $(r - 1)^2 + 1$ edges. It has girth 2 and in any vertex coloring there are either r vertices of the same color, forming a monochromatic edge, or r vertices of distinct colors, forming a rainbow edge. Assume that for any $r \geq 2$, $H(r, g - 1)$ is an rm-unavoidable hypergraph of girth at least $g - 1$.

Consider $H(r, g) = \mathcal{M} = \mathcal{M}_{a+1}$ given in the construction. Let c be a vertex coloring of \mathcal{M} . Consider the a^{th} part of $\mathcal{M} = \mathcal{M}_{a+1}$. This part corresponds to the vertex set of $\mathcal{H}_a = H(|\mathcal{M}_a|_a, g - 1)$, an rm-unavoidable hypergraph. Thus, there is a monochromatic or rainbow subset X_a in the a^{th} part of \mathcal{M} of size equal to the uniformity of \mathcal{H}_a , i.e., of size $|\mathcal{M}_a|_a$. Since $X_a \in \mathcal{E}(\mathcal{H}_a)$, X_a is the a^{th} part of a copy of \mathcal{M}_a .

Consider $(a - 1)^{\text{st}}$ part of this copy of \mathcal{M}_a . Similarly to the above, there is a monochromatic or rainbow subset X_{a-1} of this part of size equal to the uniformity of $\mathcal{H}_{a-1} = H(|\mathcal{M}_{a-1}|_{a-1}, g - 1)$, i.e., of size $|\mathcal{M}_{a-1}|_{a-1}$. Since $X_{a-1} \in \mathcal{E}(\mathcal{H}_{a-1})$, X_{a-1} is the $(a - 1)^{\text{st}}$ part of a copy of \mathcal{M}_{a-1} such that the a^{th} part of this copy is a subset of X_a .

Continuing in this manner we see that there is a monochromatic or a rainbow subset X_j of j^{th} part of \mathcal{M}_{j+1} of size equal to the uniformity of \mathcal{H}_j , i.e., of size $|\mathcal{M}_j|_j$. We have that X_j is the j^{th} part of a copy of \mathcal{M}_j such that the $(j + t)^{\text{th}}$ part of this copy is a subset of X_{j+t} , $j + t \in \{j + 1, j + 2, \dots, a\}$.

Thus X_1, X_2, \dots, X_a form parts of an a -uniform sub-hypergraph of \mathcal{M} containing a copy of \mathcal{M}_1 . Recall that \mathcal{M}_1 is a complete a -partite F -factor. Each of these parts is monochromatic or rainbow. Since $a = (r - 1)^2 + r$, there are either at least r parts that are rainbow or at least $(r - 1)^2 + 1$ parts that are monochromatic. If there are r rainbow parts, the copy of F on these parts contains a rainbow edge as F is part-rainbow-forced. So, assume that there are at least $(r - 1)^2 + 1$ monochromatic parts. If there are r of those that are of the same color, any edge in a copy of F on these parts is monochromatic. Otherwise there are at

most $(r - 1)$ parts of each given color, so there are r monochromatic parts of distinct colors. These r parts in turn contain an edge of F , and since an edge has at most one vertex from each part, this edge is rainbow.

Now, we verify that the girth of \mathcal{M} is at least g by an argument similar to one of Lemma 2. To do that, we shall prove by induction on j , that \mathcal{M}_j has girth at least g , $j = 1, \dots, a$. Since \mathcal{M}_1 is a complete a -partite F factor, it has girth equal to the girth of F , that is at least g . Assume that \mathcal{M}_j has girth at least g . Let's prove that \mathcal{M}_{j+1} has girth at least g . Recall that $\mathcal{M}_{j+1} = \mathcal{M}_j \star_j \mathcal{H}_j$, i.e., \mathcal{M}_{j+1} is obtained by amalgamating copies of \mathcal{M}_j along $\mathcal{H}_j = H(|\mathcal{M}_j|_j, g - 1)$. Let X be the j^{th} part of \mathcal{M}_{j+1} , i.e., the vertex set of \mathcal{H}_j . Consider a shortest cycle C in \mathcal{M}_{j+1} . If C is a subgraph of one of these copies of \mathcal{M}_j , then by induction C has length at least g . If the edges of C come from at least two distinct copies of \mathcal{M}_j , then C is an edge-disjoint union of hyperpaths P_0, P_1, \dots, P_{m-1} , each with at least 2 edges, from different copies of \mathcal{M}_j , such that the consecutive paths share a vertex in X , i.e., $V(P_i) \cap V(P_{i+1}) = \{u_i\}$, $i = 0, \dots, m - 1$, and u_0, \dots, u_{m-1} are distinct vertices from X , addition modulo m . Thus u_i and u_{i+1} belong to the same copy of \mathcal{M}_j and thus correspond to the vertices from the same edge of \mathcal{H}_j , $i = 0, \dots, m - 1$, addition modulo m . We see that these edges of \mathcal{H}_j form a cycle in \mathcal{H}_j of length at most half the length of C . On the other hand, we know that any cycle in \mathcal{H}_j has length at least $g - 1$, implying that C has length at least $2(g - 1) \geq g$. This concludes the proof of Theorem 1 using an explicit construction. \square

3. PROOF OF THEOREM 1 - PROBABILISTIC CONSTRUCTION

This proof is just a slight generalization of the probabilistic construction for high-girth, high-chromatic-number hypergraphs by Nešetřil and Rödl. Let an ℓ -cycle be a cycle of length ℓ . Let r, g be fixed, put $R = (r - 1)^2 + 1$ and consider an R -uniform hypergraph $\mathcal{H} = \mathcal{H}(n, R, g) = (X, \mathcal{E})$ with n vertices, girth at least g , and with $|\mathcal{E}| = \lceil n^{1+\frac{1}{g}} \rceil$. Such a graph exists, if n is large enough by Lemma 5, see Appendix.

Let's order the hyperedges of \mathcal{H} as E_1, E_2, \dots, E_m . Let \mathcal{M}_n be the family of all sequences (E'_1, \dots, E'_m) such that $|E'_i| = r$ and $E'_i \subseteq E_i$, $i = 1, \dots, m$. For a given sequence $Q \in \mathcal{M}_n$, let \mathcal{H}_Q be a hypergraph whose hyperedges are elements of Q . We say that a coloring of X is *good* for Q if there are no monochromatic and no rainbow edges under this coloring of \mathcal{H}_Q . We say that Q is colorable if there is a coloring of X that is good for Q . We shall count the number of colorable sequences and shall show that it is strictly less than the number of all sequences in \mathcal{M}_n . This will imply that there is a non-colorable sequence corresponding to an rm-unavoidable hypergraph.

Each hypergraph \mathcal{H}_Q , $Q \in \mathcal{M}_n$ has girth at least g since \mathcal{H} has this property. In addition $|\mathcal{M}_n| \geq a^{n^{1+\frac{1}{g}}}$, where $a = \binom{R}{r}$, since there are a ways to choose an r -element subset from an edge of \mathcal{H} and $m \geq n^{1+\frac{1}{g}}$. Now we consider a coloring of X with arbitrary number of colors. Each edge E of \mathcal{H} is colored with at least r or less than r colors. If E is colored with less than r colors, there are r vertices in E of the same color since E has $R = (r - 1)^2 + 1$ elements and $\frac{R}{(r-1)} > (r - 1)$. If E is colored with at least r colors, there are r vertices with pairwise

distinct colors. Thus each edge E of \mathcal{H} contains a "bad" subset that is either monochromatic or rainbow, and only at most $\binom{|E|}{r} - 1 = \binom{R}{r} - 1 = a - 1$ of all r -element subsets of E could be "good". Therefore each coloring c of X is good for at most $(a - 1)^{\lceil n^{1+\frac{1}{g}} \rceil} \leq (a - 1)^{1+n^{1+\frac{1}{g}}}$ members of \mathcal{M}_n . Since the total number of colors in X is at most n in any coloring, it is enough to consider colorings with colors $1, \dots, n$. Since there are n^n colorings with n colors we have that

$$\begin{aligned} |\{Q \in \mathcal{M}_n \mid Q \text{ is colorable}\}| &= \left| \bigcup_{c: X \rightarrow [n]} \bigcup_{Q \in \mathcal{M}_n} \{Q \mid c \text{ is good for } Q\} \right| \\ &\leq \sum_{c: X \rightarrow [n]} \left| \bigcup_{Q \in \mathcal{M}_n} \{Q \mid c \text{ is good for } Q\} \right| \\ &\leq n^n \cdot (a - 1)^{1+n^{1+\frac{1}{g}}}. \end{aligned}$$

Next we shall show that $n^n \cdot (a - 1)^{1+n^{1+\frac{1}{g}}} < a^{n^{1+\frac{1}{g}}}$ for all sufficiently large n . Indeed, $n^n(a - 1)^{1+n^{1+\frac{1}{g}}} < a^{n^{1+\frac{1}{g}}} \Leftrightarrow n \ln(n) + \ln(a - 1) < n^{1+\frac{1}{g}} \ln\left(\frac{a}{a-1}\right)$. The last inequality holds since $\ln\left(\frac{a}{a-1}\right) > 0$. Therefore the number of colorable members from \mathcal{M}_n is less than the total number of members in \mathcal{M}_n and thus there is an non-colorable $Q \in \mathcal{M}_n$ that gives \mathcal{H}_Q , an r -uniform hypergraph of girth at least g that is rm-unavoidable. \square

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4. APPENDIX

Lemma 3. *For any $\ell, g \geq 2$, $q \geq 1$ there is an ℓ -uniform hypergraph of girth at least g and minimum degree at least q .*

Proof. To see that such a hypergraph exists, consider an ℓ -uniform hypergraph F of girth at least g and chromatic number greater than q . If F has a vertex v that belongs to at most $q - 1$ edges, delete it from F . We obtain a hypergraph $F - v$ of chromatic number greater than q again because otherwise we can take a proper coloring of $F - v$ with at most q colors and extend it to a proper coloring of F . Indeed, if $E_1, \dots, E_{q'}, q' \leq q - 1$ are the edges incident to v , choose a color for v that is not a color of monochromatic $E_i - v$ under the proper coloring of $F - v$, $i = 1, \dots, q'$, if such a monochromatic edge exists. Since only at most $q - 1$ colors are forbidden for v , one color is still available. Continue this deletion process until possible. The process must stop with a non-empty graph of chromatic number greater than q and minimum degree at least q . Since it is a sub-hypergraph of the original hypergraph, it has girth at least g . \square

Lemma 4 ([16]). *Let $C(r, \ell, n)$ be the number of ℓ -cycles in the r -uniform complete hypergraph on n vertices, $r \geq 3$. Then $C(r, \ell, n) \leq c(r, \ell) \binom{n}{(r-1)\ell}$, for a function $c(r, \ell)$ independent of n .*

Proof. Observe that the largest number of vertices in an ℓ -cycle C of length ℓ is $(r - 1)\ell$. Indeed a cycle C of length ℓ is defined as a subhypergraph C with ℓ distinct vertices $x_0, \dots, x_{\ell-1}$, $\ell \geq 2$ and distinct hyperedges $E_0, \dots, E_{\ell-1}$ such that $x_i, x_{i+1} \in E_i$, $i = 0, \dots, \ell - 1$, addition of indices modulo ℓ . Thus, each hyperedge E_i , $i = 0, \dots, \ell - 1$, has at most $r - 2$ vertices not in the set $\{x_0, \dots, x_{\ell-1}\}$. Therefore the total number of vertices in C is at most $\ell(r - 2) + |\{x_0, \dots, x_{\ell-1}\}| = \ell(r - 2) + \ell = \ell(r - 1)$. Thus, an upper bound on the number of all ℓ -cycles is $\binom{n}{\ell(r-1)} \cdot c(r, \ell)$, where $\binom{n}{\ell(r-1)}$ is the number of ways to choose a set on $\ell(r - 1)$ vertices and $c(r, \ell)$ is the number of ℓ -cycles on a given set of $\ell(r - 1)$ vertices. \square

Lemma 5 ([16]). *For any positive integers r and s , $r \geq 2, s \geq 3$ there exists an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0, n \in \mathbb{N}$ there exists an r -uniform hypergraph (X, \mathcal{E}) with girth at least s and with $|\mathcal{E}| > n^{1+\frac{1}{s}}$.*

Proof. We consider a set $\mathcal{M} = \mathcal{M}(n, r, s)$ of all r -uniform hypergraphs on vertex set $[n]$ with $m = 2\lceil n^{1+1/s} \rceil$ edges. Then $|\mathcal{M}| = \binom{n}{m}$. Choose a hypergraph \mathcal{H} from \mathcal{M} randomly and uniformly, i.e., with probability $\frac{1}{|\mathcal{M}|}$. Let K be a complete r -uniform hypergraph on vertex set $[n]$. Call cycles of length smaller than s bad. Let X_j be the number of cycles of length j in \mathcal{H} and X_{bad} be the number of bad cycles. Then $\text{Exp}(X_j) = \sum_C \text{Prob}(C \subseteq \mathcal{H})$, where the sum is over all cycles C of length j in K . Then $\text{Exp}(X_j) \leq C(r, j, n) \frac{\binom{n}{r-j}}{\binom{n}{r}}$, where $C(r, j, n)$ is the number of cycles of length j in K and second term is the probability of occurrence of such a cycle. Using Lemma 4, we have that $\text{Exp}(X_j) \leq c(r, j) \binom{n}{(r-1)j} \frac{\binom{n}{m-j}}{\binom{n}{m}}$. Then, for constants $\tilde{c}(r, j)$, $j = 2, \dots, s-2$ and $\tilde{C}(r, s)$, we have

$$\begin{aligned}
\text{Exp}(X_{bad}) &= \sum_{j=2}^{s-1} \text{Exp}(X_j) \\
&\leq \sum_{j=2}^{s-1} c(r, j) \cdot \binom{n}{(r-1)j} \frac{\binom{n}{m-j}}{\binom{n}{m}} \\
&= \sum_{j=2}^{s-1} c(r, j) \cdot \binom{n}{(r-1)j} \frac{m \cdot (m-1) \cdots (m-j+1)}{\binom{n}{r} \cdot ((\binom{n}{r}) - 1) \cdots ((\binom{n}{r}) - j + 1)} \\
&\leq \sum_{j=2}^{s-1} c(r, j) \cdot \binom{n}{(r-1)j} \left(\frac{m}{\binom{n}{r}} \right)^j \\
&\leq \sum_{j=2}^{s-1} \tilde{c}(r, j) n^{(r-1)j-rj} m^j \\
&\leq \sum_{j=2}^{s-1} \tilde{c}(r, j) n^{(r-1)j-rj} n^{(1+1/s)j} \\
&\leq \tilde{C}(r, s) n.
\end{aligned}$$

Since $\text{Exp}(X_{bad}) \leq \tilde{C}(r, s)n$, there is a hypergraph from \mathcal{M} with at most $\tilde{C}(r, s)n$ cycles of length at most $s-1$. Delete an edge from each such cycle and obtain a hypergraph on at least $2n^{1+1/s} - \tilde{C}(r, s)n > n^{1+1/s}$ edges and girth at least s . \square

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